

Radial basis function collocation method for an elliptic problem with nonlocal multipoint boundary condition

Svajūnas Sajavičius*

*Department of Computer Science II, Faculty of Mathematics and Informatics,
Vilnius University, Naugarduko str. 24, LT-03225 Vilnius, Lithuania*

*Department of Finance and Taxes, Faculty of Economics and Finance Management,
Mykolas Romeris University, Ateities str. 20, LT-08303 Vilnius, Lithuania*

Abstract

Radial basis function domain-type collocation method is applied for an elliptic partial differential equation with nonlocal multipoint boundary condition. A geometrically flexible meshless framework is suitable for imposing nonclassical boundary conditions which relate the values of unknown function on the boundary to its values at a discrete set of interior points. Some properties of the method are investigated by a numerical study of a test problem with the manufactured solution. Attention is mainly focused on the influence of nonlocal boundary condition. The standard collocation and least squares approaches are compared. In addition to its geometrical flexibility, the examined method seems to be less restrictive with respect to parameters of nonlocal conditions than, for example, methods based on finite differences.

Keywords: elliptic problem, nonlocal multipoint boundary condition, meshless method, radial basis function, collocation, least squares

1. Introduction

Mathematical models arising in various fields of science and engineering (for example, thermoelasticity [1], thermodynamics [2], hydrodynamics [3], biological fluid dynamics [4] or plasma physics [5]) are very often expressed in terms of partial differential equations (PDEs) and nonclassical constraints, which are usually identified as *nonlocal (boundary) conditions*. As a rule, the appearance of nonlocal conditions makes quite a number of theoretical and numerical challenges. Therefore, nonlocal differential problems receive a lot of attention in the literature. Many studies are dedicated to various aspects of the numerical solution of such nonclassical problems.

Finite differences, finite elements or finite volumes are examples of discretisation techniques which are widely and successfully applied to approximate solution of various PDEs. These traditional approaches are based on the domain discretisation using mesh. The mesh generation can be quite a difficult task in case of three-dimensional domains with complex shapes. One of the ways to overcome problems related to the meshing are so-called *meshless methods*, which gained a lot of attention in recent years (see e.g. [6–8]).

In this paper, for the solution of a model problem with nonlocal boundary condition we apply a meshless discretisation technique based the *radial basis functions* (RBFs) [9–14]. In papers [15–19], RBFs were already used for the spatial discretisation of time dependent (parabolic and hyperbolic) equations with nonlocal integral conditions. Experiments with various test examples have demonstrated that RBF-based collocation methods can be successfully applied to solve such a kind of nonlocal problems. Recently, the method of approximate particular solutions using multiquadric (MQ) RBFs has been applied to the time-fractional diffusion equation with nonlocal boundary condition [20].

It is well-known that properties of the numerical methods for nonlocal differential problems usually depend on the parameters of nonlocal boundary conditions. For example, the stability of finite difference schemes are related to the spectral properties of certain matrices and, in case of nonlocal problems, these properties depend on the parameters appearing in nonlocal conditions [21–23]. Therefore, numerical methods for the solution of PDEs with nonlocal conditions require special attention. The influence of nonlocal conditions on the properties of any numerical method which is applied to solve a certain nonlocal problem should always be investigated very carefully.

In paper [24], RBF collocation technique was applied to solve a model problem for two-dimensional Poisson equation on the unit rectangle. Two-point or integral condition was formulated on one side of the rectangle. Besides the standard testing of the method, the influence of nonlocal conditions on the optimal selection of the RBF shape parameter, as well as on

*Correspondence address: Department of Computer Science II, Faculty of Mathematics and Informatics, Vilnius University, Naugarduko str. 24, LT-03225 Vilnius, Lithuania. Tel.: +370 5 2193091; fax: +370 5 2151585.

Email address: svajunas.sajavicius@mif.vu.lt
svajunas@mr.uni.eu (Svajūnas Sajavičius)

URL: <http://www.mif.vu.lt/~svajunas>
<http://bit.ly/svajunas> (Svajūnas Sajavičius)

the conditioning and accuracy of the method was investigated. Later, the same problem has been solved in paper [25], where collocation methods based on Haar wavelets and RBFs have been applied. RBF-based collocation technique was also used to solve a multidimensional elliptic equation with nonlocal integral conditions [26]. The influence of the RBF shape parameter and distribution of the nodes on the accuracy of the method as well as the influence of nonlocal conditions on the conditioning of the collocation matrix were investigated by analysing two- and three-dimensional test problems with the manufactured solutions.

In the present work we continue our investigation and consider a model problem which consists of Poisson equation with mixed boundary conditions. One of these conditions is *nonlocal multipoint boundary condition* relating the boundary values of unknown function to several values inside the domain. A meshless method for the solution of such problem allows to eliminate connection between the domain discretisation and points defining nonlocal part of the multipoint boundary condition. To the best of our knowledge, this is the first time when an elliptic PDE with nonlocal multipoint boundary condition is solved using an RBF-based meshless method.

The main aim of this work is to investigate the properties of the method and, in particular, their dependence on the parameters of nonlocal condition. We do not provide any theoretical results. Instead, we conduct an extensive numerical study. The insights made from the numerical study can help us gain the basic understanding of the properties of the method. It should be mentioned that, when dealing with nonlocal problems, numerical studies (computational experiments) are often utilised as research methods even when such classical and well-established techniques as finite differences are applied (see e.g. [22, 27]).

The paper is organised as follows. In Section 2, we give a detailed formulation of the model problem and some additional related references. A meshless method based on RBF collocation is described in Section 3. By analysing a test example, the method is investigated in Section 4. Finally, Section 5 concludes the paper with summarising remarks and possible directions for the future research.

2. Model problem

A model problem considered in this paper consists of Poisson equation and mixed boundary conditions:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, & (1a) \\ u = g & \text{on } \Gamma_1, & (1b) \\ u = \sum_{\mathbf{x}_l^* \in \Omega^*} \gamma_l u(\mathbf{x}_l^*) + h & \text{on } \Gamma_2, & (1c) \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) is a bounded domain, $\partial\Omega = \Gamma_1 \cup \Gamma_2$ (with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\Gamma_2 \neq \emptyset$), f , g and h are given functions. While Dirichlet condition (1b) is an example of classical boundary condition, the condition (1c) is nonlocal and relates the values of the solution u on the boundary part Γ_2 to the values at the interior points $\mathbf{x}_l^* \in \Omega^* \subset \Omega$ (subset Ω^* is discrete). The

nonlocal condition (1c) is defined by parameters γ_l and \mathbf{x}_l^* . The weights γ_l can be either constants, or functions ($\gamma_l = \gamma_l(\mathbf{x})$), or values of the functions at $\mathbf{x}_l^* \in \Omega^*$ ($\gamma_l = \gamma_l(\mathbf{x}_l^*)$). When $\gamma_l \equiv 0$, condition (1c) becomes Dirichlet boundary condition. If condition (1c) relates the boundary values to a single interior point ($|\Omega^*| = 1$), it is usually referenced to as *Bitsadze–Samarskii nonlocal condition*.

Nonlocal multipoint boundary conditions are related to nonlocal integral conditions [21]. For instance, the multipoint condition (1c) is a special case of nonlocal integral condition

$$u = \int_{\Omega} \gamma(\mathbf{x}) u(\mathbf{x}) d\mathbf{x} + h \quad \text{on } \Gamma_2.$$

Indeed, we get the nonlocal multipoint boundary condition (1c) when the weight function is defined as

$$\gamma(\mathbf{x}) = \sum_{\mathbf{x}_l^* \in \Omega^*} \gamma_l \delta(\|\mathbf{x} - \mathbf{x}_l^*\|_2),$$

where δ is the Dirac delta function.

Numerical methods for the solution of PDEs with nonlocal discrete boundary conditions have been considered in various papers. Usually, such numerical methods are based on finite differences. For example, we can mention papers related to the approximate solution of nonlocal problems for elliptic [28–31], elliptic–parabolic [32], hyperbolic [33, 34], or hyperbolic–parabolic [35] equations with multipoint or Bitsadze–Samarskii nonlocal boundary conditions. The paper [36] presents an efficient way of implementing general multipoint constraint conditions arising in finite element analysis related to structural mechanics. In paper [37], a two-dimensional reaction–diffusion problem with Bitsadze–Samarskii nonlocal boundary condition was solved using meshless local Petrov–Galerkin (MLPG) method and MQ RBFs were used for the spatial discretisation of local weak equations.

3. Construction of the method

3.1. A brief overview of radial basis functions

RBFs already proved to be quite an effective tool both for the scattered data interpolation [10] and approximate solution of PDEs [11–14]. We give a very brief introduction to RBFs. More details can be found in books [9, 10, 12, 14]. The book [12] also reviews the latest advances on RBF collocation methods for the numerical solution of PDEs.

A multivariate real-valued function $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is called a *radial function* if there exists a univariate function $\phi : [0, \infty) \rightarrow \mathbb{R}$ such that

$$\Phi(\mathbf{x}) = \phi(\|\mathbf{x}\|),$$

where $\|\cdot\|$ is some norm on \mathbb{R}^d (usually the Euclidean norm is used). That is, the function $\Phi(\mathbf{x})$ can be expressed in the Euclidean distance variable $r = \|\mathbf{x}\|$.

Table 1 gives several examples of widely used RBFs. All these RBFs are globally supported and infinitely smooth. The shape (flatness) of each given RBF is controlled by a positive

parameter ϵ which is called the *shape parameter*. Equivalently, a reciprocal shape parameter $c = 1/\epsilon$ also can be used in the expressions of the same functions.

In the case of an interpolation problem, the positive definiteness of RBF is an important property which ensures the invertibility of the interpolation matrix. IMQ, IQ and GA RBFs are strictly positive definite, while MQ RBF is conditionally positive definite of order one [9, 10]. The conditional positive definiteness means that the invertibility of the interpolation problem is ensured by adding a polynomial of a certain order to the interpolant and by augmenting the interpolation system with some additional equations. Since the polynomial augmentation can increase the condition number of the interpolant matrix but not necessary the accuracy of the interpolation [38], and singular cases of the interpolation matrix almost never appear in practice, we will construct and investigate the method without such modification.

RBF-based collocation methods can be categorised as domain-type or boundary-type methods [12]. The Kansa method applied in this paper is an example of domain-type methods, while the method of fundamental solutions (MFS) is one of well-known boundary-type methods.

3.2. Representation of the domain

Instead of meshing, which is essential procedure, for example, in finite element method, meshless methods require only domain representation by nodes which can be scattered in unstructured fashion. We use the set of nodes (*collocation points*)

$$\Xi = \{\mathbf{x}_i\}_{i=1}^N \subset \bar{\Omega},$$

which consists of three pairwise disjoint subsets representing domain Ω and its boundary parts Γ_1 and Γ_2 :

$$\Xi_\Omega = \{\mathbf{x} \in \Xi : \mathbf{x} \in \Omega\} \subset \Omega,$$

$$\Xi_{\Gamma_1} = \{\mathbf{x} \in \Xi : \mathbf{x} \in \Gamma_1\} \subset \Gamma_1,$$

$$\Xi_{\Gamma_2} = \{\mathbf{x} \in \Xi : \mathbf{x} \in \Gamma_2\} \subset \Gamma_2.$$

We assume that $\Xi = \Xi_\Omega \cup \Xi_{\Gamma_1} \cup \Xi_{\Gamma_2}$, $\Xi_\Omega \neq \emptyset$, $\Xi_{\Gamma_1} \neq \emptyset$ (if $\Gamma_1 \neq \emptyset$) and $\Xi_{\Gamma_2} \neq \emptyset$. We also assume that $\Omega^* \subset \Xi_\Omega$, i.e., points \mathbf{x}_i^* defining nonlocal boundary condition (1c) are included in the domain representation.

When differential problems with nonlocal multipoint boundary conditions are solved using finite differences, usually it is assumed that the points \mathbf{x}_i^* coincide with some points of the grid (see e.g. [21]). Such assumption simplifies theoretical analysis of the numerical scheme [39] but requires certain modifications.

Table 1
Globally supported and infinitely smooth RBFs ($r = \|\mathbf{x}\|$).

RBF	Definition
Multiquadric (MQ)	$\phi(r) = \sqrt{1 + (\epsilon r)^2}$
Inverse Multiquadric (IMQ)	$\phi(r) = (\sqrt{1 + (\epsilon r)^2})^{-1}$
Inverse Quadric (IQ)	$\phi(r) = (1 + (\epsilon r)^2)^{-1}$
Gaussian (GA)	$\phi(r) = \exp(-(\epsilon r)^2)$

In order to impose nonlocal multipoint boundary condition similar to (1c) in a mesh-based method, we need either to ensure that the points \mathbf{x}_i^* also would be the nodes of the mesh, or to approximate these points by substituting them with closely located nodes of the mesh. The first approach requires additional mesh modifications, while the later can decrease the accuracy of the results, especially when coarse mesh is used. Here geometrical flexibility of meshless methods comes in handy. In case of meshless method, nonlocal multipoint boundary conditions can be imposed without any additional treatment. It is enough to ensure that all points $\mathbf{x}_i^* \in \Omega^*$ belong to the set Ξ .

3.3. Collocation in the least squares mode

The collocation points Ξ not necessary should coincide with the RBF centers (*source points*). We consider the case when the set of RBF centers Ξ' is a subset of the set of collocation points. Thus, we assume that

$$\Xi' = \{\mathbf{x}'_i\}_{i=1}^M \subseteq \Xi,$$

where $\Xi' = \Xi'_\Omega \cup \Xi'_{\Gamma_1} \cup \Xi'_{\Gamma_2}$ with $\Xi'_\Omega \subseteq \Xi_\Omega$, $\Xi'_{\Gamma_1} \subseteq \Xi_{\Gamma_1}$, $\Xi'_{\Gamma_2} \subseteq \Xi_{\Gamma_2}$, and $|\Xi'| = M \leq N$.

As usually, we seek for an approximate solution to the problem (1) in the form

$$\bar{u}(\mathbf{x}) = \boldsymbol{\phi}^T(\mathbf{x}) \boldsymbol{\lambda},$$

where

$$\boldsymbol{\phi}(\mathbf{x}) = [\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \dots, \phi_M(\mathbf{x})]^T$$

and

$$\phi_i(\mathbf{x}) = \phi(\|\mathbf{x} - \mathbf{x}'_i\|_2)$$

for a given RBF ϕ and a source point $\mathbf{x}'_i \in \Xi'$, $\|\cdot\|_2$ denotes the Euclidean norm, and $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \dots, \lambda_M]^T$ are the coefficients to be determined using collocation approach.

By requiring approximate solution $\bar{u}(\mathbf{x})$ to satisfy Poisson equation (1a) on the nodes representing Ω we get the following linear equations:

$$-\Delta \boldsymbol{\phi}^T(\mathbf{x}) \boldsymbol{\lambda} = f(\mathbf{x}), \quad \mathbf{x} \in \Xi_\Omega. \quad (2a)$$

From Dirichlet boundary condition (1b) (if $\Gamma_1 \neq \emptyset$) we obtain equations

$$\boldsymbol{\phi}^T(\mathbf{x}) \boldsymbol{\lambda} = g(\mathbf{x}), \quad \mathbf{x} \in \Xi_{\Gamma_1}, \quad (2b)$$

and discretisation of nonlocal multipoint boundary condition (1c) leads to

$$\left(\boldsymbol{\phi}^T(\mathbf{x}) - \sum_{\mathbf{x}_j \in \Omega} \gamma_j \boldsymbol{\phi}^T(\mathbf{x}_j) \right) \boldsymbol{\lambda} = h(\mathbf{x}), \quad \mathbf{x} \in \Xi_{\Gamma_2}. \quad (2c)$$

Thus, in order to determine coefficients $\boldsymbol{\lambda}$, we need to solve the linear system (2):

$$\mathbf{A} \boldsymbol{\lambda} = \mathbf{b}, \quad (3)$$

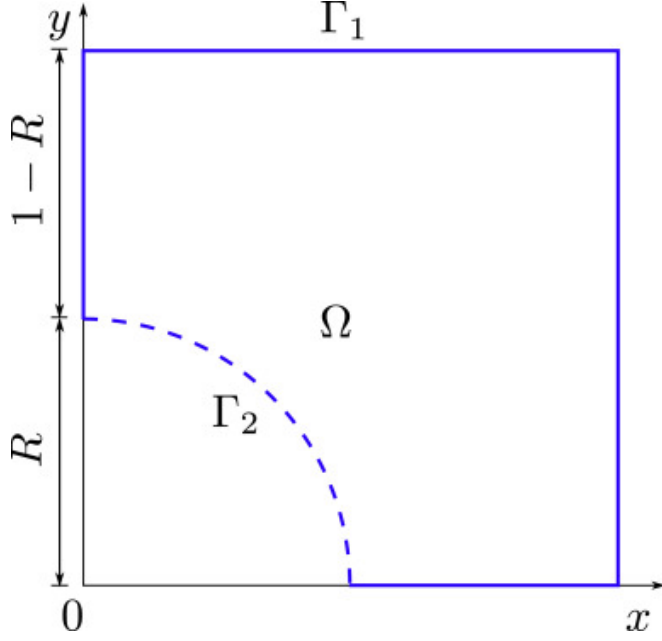


Figure 1: Two-dimensional problem description. The boundary parts Γ_1 and Γ_2 are depicted as solid and dashed lines, respectively.

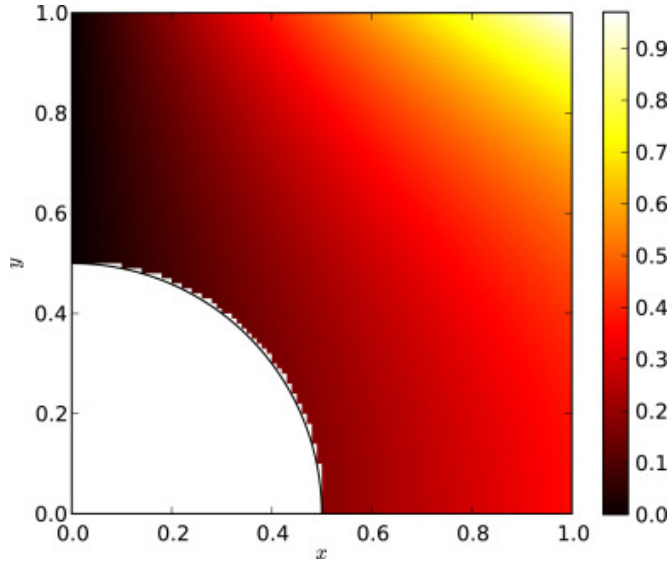


Figure 2: The exact solution of the test problem.

where \mathbf{A} is the *collocation matrix* and \mathbf{b} is the right-hand side vector. If $M < N$, the system (3) is overdetermined and we can treat it as a linear least squares problem. The solution of this problem is

$$\hat{\lambda} = \arg \min_{\lambda \in \mathbb{R}^M} \|\mathbf{A}\lambda - \mathbf{b}\|_2.$$

It is known that matrix $(\mathbf{A})_{N \times M}$ tends to be ill-conditioned as $\epsilon \rightarrow 0$. That is probably the main drawback of all collocation methods based on globally supported RBFs. The influence of the shape parameter will be illustrated and some properties of the method will be investigated in the numerical study.

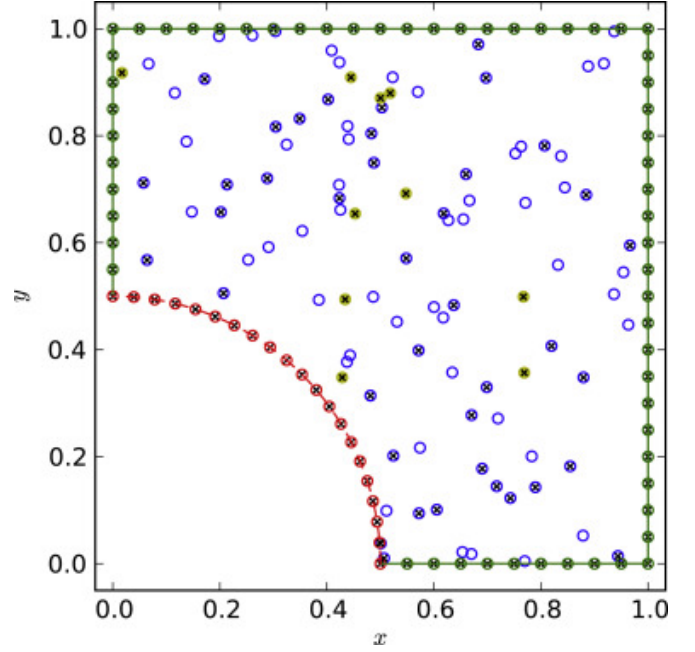


Figure 3: Representation of the domain Ω and its boundary $\partial\Omega$: $R = 0.5$, $|\Xi_\Omega| = 100$, $|\Xi'_\Omega| = 50$, $|\Xi_{\Gamma_1}| = 59$, $|\Xi_{\Gamma_2}| = 21$, $|\Omega^*| = 10$, source points are denoted using crossed circles and points of the set Ω^* are filled.

4. Numerical study

4.1. Technical details

A two-dimensional ($d = 2$) problem with the manufactured solution was analysed in order to demonstrate the applicability and efficiency of the method, as well as to investigate some of its properties. The problem was formulated on the unit rectangle with a cut of the radius R (see Fig. 1). Three different cases (examples) of the weights γ_l were considered:

Case 1: γ_l are constants: $\gamma_l \equiv \gamma$, γ is a constant;

Case 2: γ_l are functions: $\gamma_l = \gamma_l(\mathbf{x}) \equiv \gamma \cdot \|(1, 0) - \mathbf{x}\|_2$;

Case 3: γ_l are values of the functions at $\mathbf{x}_l^* \in \Omega^*$: $\gamma_l = \gamma_l(\mathbf{x}_l^*) \equiv \gamma \cdot \|\mathbf{x}_l^*\|_2$.

In all cases, the initial data was chosen so that the exact solution of the problem would be the function

$$u(\mathbf{x}) = x \exp(y^2 - 1), \quad \mathbf{x} = (x, y).$$

In Fig. 2, this solution is depicted on the domain with the cut of the radius $R = 0.5$.

The representation of the domain Ω ($R = 0.5$) which was used in the study is depicted in Fig. 3. The boundary parts Γ_1 and Γ_2 were represented by regularly distributed nodes, while the nodes representing Ω were scattered completely randomly. The set of source points used in the least squares mode was

$$\Xi' = \Xi_{\Gamma_1} \cup \Xi_{\Gamma_2} \cup \Xi'_{\Omega},$$

where Ξ'_{Ω} is consisted of nodes randomly selected from Ξ_Ω (see Fig. 3). The same representation of the domain as well as given

distribution of the source points was used in the entire numerical study.

The distribution of the points $\mathbf{x}_l^* \in \Omega^*$ on the domain Ω was fixed everywhere except the experiment in which the influence of this distribution was investigated. Also we assumed that $\gamma_l \equiv 1$ by default, and different values of γ_l were used only when the influence of these parameters was investigated.

The accuracy of the method can be estimated using various error measures such as the normalised L^2 -error

$$E_{L^2} = \sqrt{\frac{\sum_{\mathbf{x} \in \Xi_{\text{test}}} (u(\mathbf{x}) - \bar{u}(\mathbf{x}))^2}{\sum_{\mathbf{x} \in \Xi_{\text{test}}} u^2(\mathbf{x})}}.$$

where $\Xi_{\text{test}} \subset \bar{\Omega}$ is a set of test points. In our numerical study, the accuracy was estimated on the collocation nodes ($\Xi_{\text{test}} = \Xi$). The conditioning of the matrix \mathbf{A} was evaluated using the traditional condition number

$$\kappa(\mathbf{A}) = \frac{\sigma_{\max}}{\sigma_{\min}},$$

where σ_{\max} and σ_{\min} are the maximal and minimal singular values of the matrix.

The method has been implemented in PYTHON programming language using SciPy package. The linear problem (3) was solved using `lstsq` routine from `LINALG` subpackage [40]. This routine was used to solve both the standard collocation system ($M = N$) and the least squares problem ($M < N$). In comparison with the routine `solve`, which was used in our previous studies [24, 26], the routine `lstsq` allows to solve the linear problems obtained by using smaller values of the shape parameter ϵ . A useful analysis of direct solvers (available both in MATLAB and PYTHON environments) applied to linear systems arising from the interpolation and the approximation of PDEs using RBFs is given in [41].

The presented results were obtained using GA RBFs with the shape parameter $\epsilon = 2.0$ and $\gamma \equiv 1$, unless mentioned otherwise.

4.2. Results and discussion

4.2.1. Selection of the shape parameter

Successful selection of the RBF shape parameter ϵ is of great importance to the accuracy of the numerical results. Small values of the shape parameter are required in order to get accurate results. On the other hand, the method becomes useless due to ill-conditioning if this parameter is too small. Here we deal with the so-called *uncertainty* (or *trade-off*) *principle* [10, 42].

There exist various techniques for the optimal selection of the shape parameter (see e.g. [43–46]). The multiple criteria decision making techniques have been applied in paper [47]. Recently, hybrid and binary shape parameter selection strategies have been proposed and investigated [48]. However, this question is not completely answered and successful selection of the shape parameter is one of the most challenging problems related to the application of RBF-based methods.

To illustrate the importance of the shape parameter ϵ , we have solved the examined test problem with different values of

this shape parameter. We also compared results obtained using the standard collocation approach and the method in the least squares mode.

From Fig. 4 we see that the method in the least squares mode has better conditioning but lower accuracy. The same figure allows to indicate the range of such values of the shape parameters which allow to obtain results of a reasonable accuracy. Once again we emphasise that the problem of the optimal selection of the shape parameter is open. However, our previous study [24] indicates that the variability of the optimal shape parameters with respect to parameters of nonlocal conditions is usually low. This allows us to expect that if the shape parameter could be successfully selected for a problem with classical boundary conditions, the same value of the shape parameter could be used when solving a nonlocal problem.

4.2.2. Accuracy of approximation of nonlocal boundary condition

Nonlocal boundary conditions such as (1c) reflect physical situations in which direct measurements of the data on the domain boundary are impossible, or the data on the boundary depend on the data inside the domain. Besides the examination of the general accuracy of the approximate solution obtained using the considered method, we also investigated how accurately this method can recover the unknown solution on the boundary of the domain.

Fig. 5 presents the absolute errors of the obtained approximate solutions. From this figure we observe that it is important to represent the domain by nodes distributed as uniformly as possible. The peaks of the errors are located in the areas which lack collocation points.

In Fig. 6 we see the absolute errors on the boundary Γ_2 where the solution is defined by nonlocal multipoint boundary condition (1c). The standard collocation gives smaller variance in the absolute error calculated on the boundary Γ_2 .

4.2.3. Influence of nonlocal boundary condition on the properties of the method

Nonlocal boundary conditions usually have strong influence on the properties of the numerical method which is applied to solve a nonlocal differential problem. In comparison to classical cases, theoretical investigations of such methods become more difficult and require applying various nonstandard techniques. For example, stability analysis of finite difference schemes for PDEs with nonlocal boundary conditions requires to investigate the spectrum of special non-symmetric matrices (see e.g. [21]). Such investigations sometimes can be so challenging that classical analytical methods are not enough to answer all the questions [22, 27].

First of all, we investigated how the location of the points \mathbf{x}_l^* defining nonlocal boundary condition (1c) affects the properties of the method (its conditioning and accuracy). We used exactly the same domain representation which is given in Fig. 3. Points \mathbf{x}_l^* were randomly selected from the set Ξ'_Ω ($|\Omega^*| = 10$). The weight γ in all three cases was assumed to be equal to 1. The sample of 1000 such trials was generated and analysed.

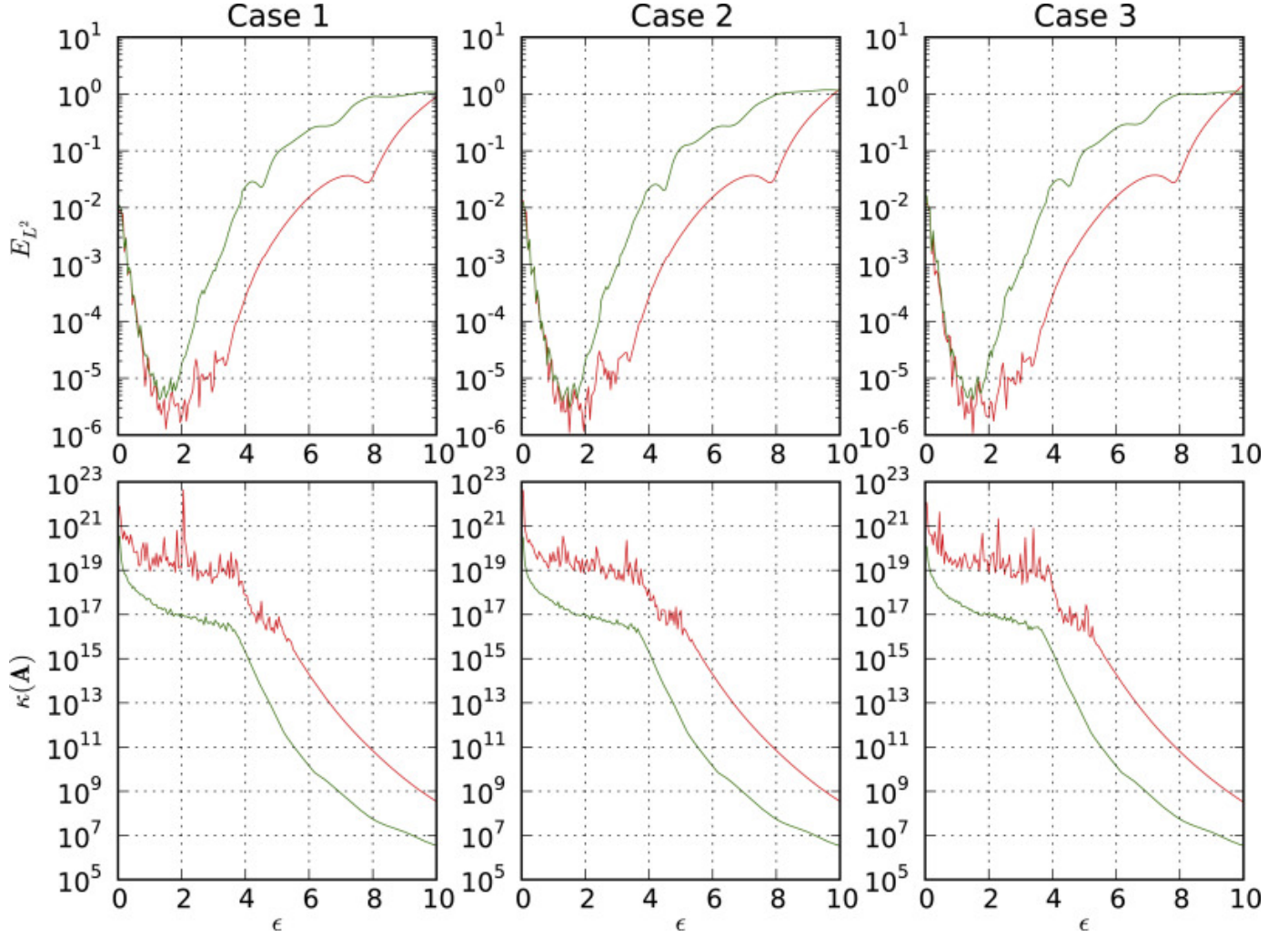


Figure 4: The dependence of the accuracy and conditioning of the method in the standard collocation (red curves) and least squares (green curves) modes on the values of the GA RBF shape parameter. (For interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

In Fig. 7 the results of this experiment are depicted, while in Table 2 the descriptive statistics (the minimal and maximal values, quartiles, means, standard deviations and coefficients of variance of the normalised L^2 -errors and condition numbers $\kappa(\mathbf{A})$) is reported. We were mainly interested in the variability of the accuracy with respect to the location of points \mathbf{x}_l^* . The variability is measured by the standard deviation or by the coefficient of variance (the ratio of the standard deviation to the mean). From Table 2 we can see that the coefficients of variance are smaller than one in all the cases, i.e. the variability of the errors with respect to the distribution of points \mathbf{x}_l^* is low. Moreover, in all three cases the variability in errors E_{L^2} , contrary to condition numbers $\kappa(\mathbf{A})$, is lower when the standard collocation approach is applied.

We also investigated the influence of the weights γ_l . Under assumption that all the weights $\gamma_l \equiv \gamma$, we solved the test problem with various values of γ . Representation of the domain and location of points \mathbf{x}_l^* were fixed (see Fig. 3).

In the considered test problem, nonlocal boundary condition (1c) is defined on a relatively small part of the boundary

$\partial\Omega$. However, from Fig. 8 we see that this condition has quite strong negative influence on the conditioning of the matrix \mathbf{A} . It is clear that the conditioning of the method tends to deteriorate as $|\gamma|$ increases. Despite of this, we can achieve quite accurate results in a wide range of positive or negative values of γ .

5. Concluding remarks

We have constructed a meshless method for the solution of an elliptic boundary value problem involving nonlocal multipoint boundary condition. The method is based on the collocation of RBFs. By analysing a two-dimensional test problem, we demonstrated the applicability and efficiency of the method. Geometrical flexibility of the method allows us to impose nonlocal multipoint boundary conditions very easily. We considered only a two-dimensional test example but RBF-based meshless methods can be extended for multidimensional problems without any significant additional effort. While the method has disadvantages which are quite common between RBF-based methods (sensitivity to the shape parameter, ill-condi-

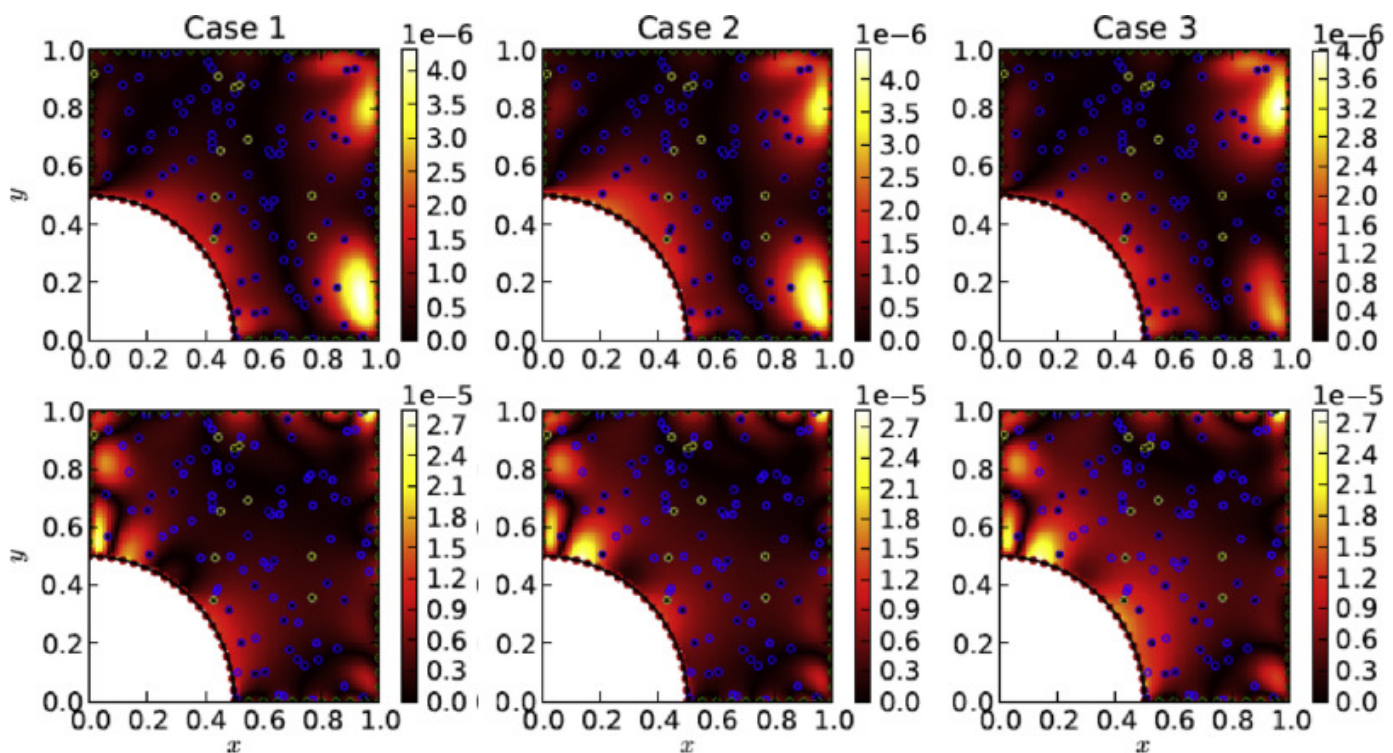


Figure 5: The absolute errors $|u(x, y) - \bar{u}(x, y)|$ obtained using the method in the standard collocation (top row) and least squares (bottom row) modes (the collocation and source points are denoted as in Fig. 3).

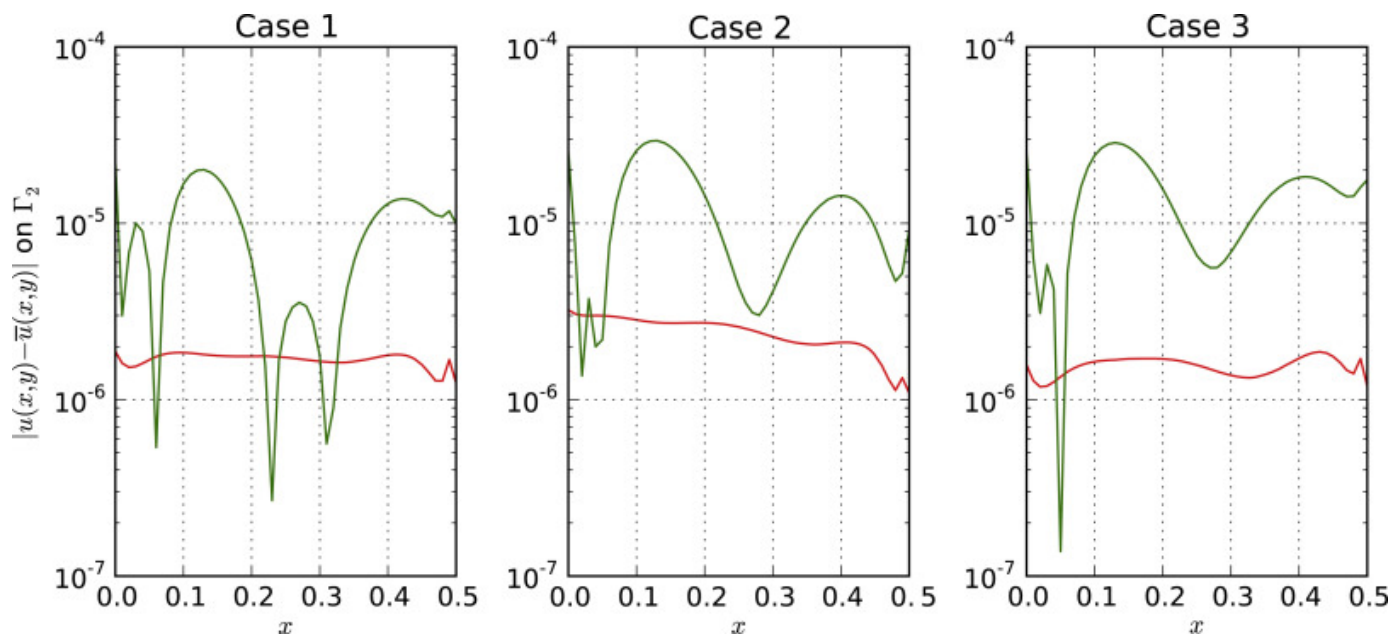


Figure 6: The absolute errors $|u(x, y) - \bar{u}(x, y)|$ obtained using the method in the standard collocation (red curves) and least squares (green curves) modes on the boundary part Γ_2 (points $(x, y) = (x, \sqrt{R^2 - x^2})$). (For interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

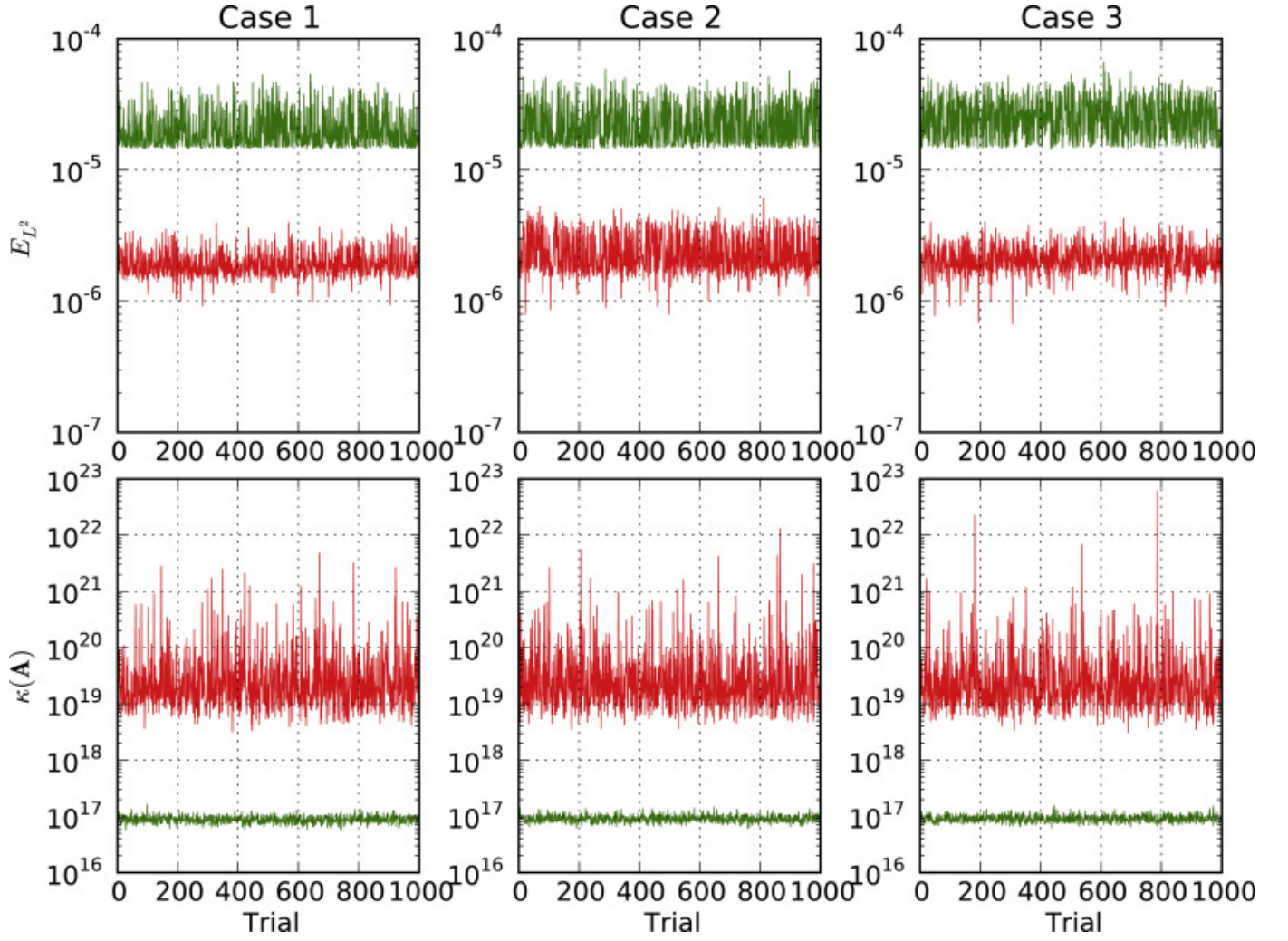


Figure 7: The errors E_{L_2} and condition numbers $\kappa(\mathbf{A})$ obtained applying the method in the standard collocation (red curves) and least squares (green curves) modes for the test problem with 1000 different distributions of points \mathbf{x}_T^* . (For interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

Table 2: Descriptive statistics of the errors E_{L_2} and condition numbers $\kappa(\mathbf{A})$ obtained applying the method for the test problem with 1000 different distributions of points \mathbf{x}_T^* .

	Case 1		Case 2		Case 3	
	E_{L_2}	$\kappa(\mathbf{A})$	E_{L_2}	$\kappa(\mathbf{A})$	E_{L_2}	$\kappa(\mathbf{A})$
<i>Standard collocation</i>						
Min.	$9.23903063 \times 10^{-7}$	$3.18749387 \times 10^{18}$	$7.92397600 \times 10^{-7}$	$3.54020212 \times 10^{18}$	$6.74089780 \times 10^{-7}$	$3.08649045 \times 10^{18}$
Q_1	$1.61880912 \times 10^{-6}$	$1.06516023 \times 10^{19}$	$1.73377778 \times 10^{-6}$	$1.07524772 \times 10^{19}$	$1.77451788 \times 10^{-6}$	$1.05842896 \times 10^{19}$
Q_2 (median)	$1.80608282 \times 10^{-6}$	$1.85106654 \times 10^{19}$	$2.08031674 \times 10^{-6}$	$1.90036992 \times 10^{19}$	$2.04331856 \times 10^{-6}$	$1.89319852 \times 10^{19}$
Q_3	$2.13492762 \times 10^{-6}$	$3.99630073 \times 10^{19}$	$3.03098088 \times 10^{-6}$	$4.21979870 \times 10^{19}$	$2.46350102 \times 10^{-6}$	$3.97811914 \times 10^{19}$
Max.	$3.97150860 \times 10^{-6}$	$4.77346066 \times 10^{21}$	$6.05333836 \times 10^{-6}$	$1.30464709 \times 10^{22}$	$4.26324657 \times 10^{-6}$	$6.02673054 \times 10^{22}$
Mean	$1.94105748 \times 10^{-6}$	$6.75872754 \times 10^{19}$	$2.41986521 \times 10^{-6}$	$8.66988466 \times 10^{19}$	$2.14058392 \times 10^{-6}$	$1.41517304 \times 10^{20}$
Standard deviation	$4.64836023 \times 10^{-7}$	$2.66588695 \times 10^{20}$	$8.77979417 \times 10^{-7}$	$5.17670752 \times 10^{20}$	$5.30983206 \times 10^{-7}$	$2.04657717 \times 10^{21}$
Coefficient of variance	$2.39475660 \times 10^{-1}$	3.94436221×10^0	$3.62821620 \times 10^{-1}$	5.97090703×10^0	$2.48055309 \times 10^{-1}$	1.44616743×10^1
<i>Least squares mode</i>						
Min.	$1.44330461 \times 10^{-5}$	$5.79845300 \times 10^{16}$	$1.44235194 \times 10^{-5}$	$6.13269051 \times 10^{16}$	$1.43643690 \times 10^{-5}$	$6.31777260 \times 10^{16}$
Q_1	$1.54616742 \times 10^{-5}$	$7.86434211 \times 10^{16}$	$1.59560175 \times 10^{-5}$	$8.21110146 \times 10^{16}$	$1.82322409 \times 10^{-5}$	$8.24833076 \times 10^{16}$
Q_2 (median)	$1.77345882 \times 10^{-5}$	$8.68939671 \times 10^{16}$	$1.96894301 \times 10^{-5}$	$8.94004911 \times 10^{16}$	$2.49316895 \times 10^{-5}$	$9.00126013 \times 10^{16}$
Q_3	$2.44841949 \times 10^{-5}$	$9.65011843 \times 10^{16}$	$2.92639628 \times 10^{-5}$	$9.86839655 \times 10^{16}$	$3.36449547 \times 10^{-5}$	$9.94490915 \times 10^{16}$
Max.	$5.35494299 \times 10^{-5}$	$1.60091154 \times 10^{17}$	$5.86708958 \times 10^{-5}$	$1.46041394 \times 10^{17}$	$6.60254533 \times 10^{-5}$	$1.55122388 \times 10^{17}$
Mean	$2.10635962 \times 10^{-5}$	$8.81422848 \times 10^{16}$	$2.34063988 \times 10^{-5}$	$9.13247640 \times 10^{16}$	$2.65097147 \times 10^{-5}$	$9.17334058 \times 10^{16}$
Standard deviation	$7.56511156 \times 10^{-6}$	$1.31330873 \times 10^{16}$	$9.09743069 \times 10^{-6}$	$1.32619260 \times 10^{16}$	$9.36260749 \times 10^{-6}$	$1.32512543 \times 10^{16}$
Coefficient of variance	$3.59155744 \times 10^{-1}$	$1.48998716 \times 10^{-1}$	$3.88672806 \times 10^{-1}$	$1.45217194 \times 10^{-1}$	$3.53176471 \times 10^{-1}$	$1.44453966 \times 10^{-1}$

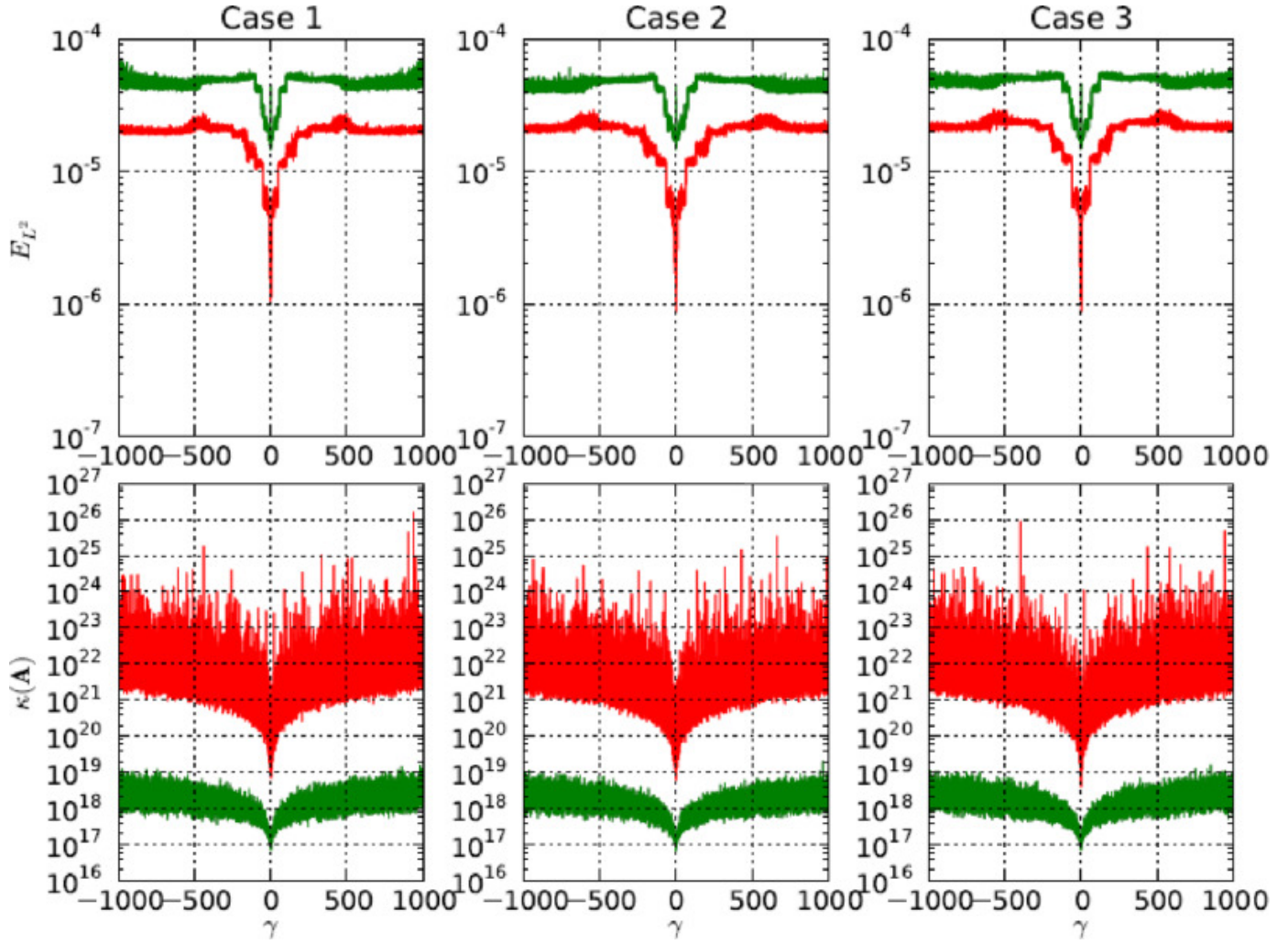


Figure 8: The dependence of the accuracy and conditioning of the method in the standard collocation (red curves) and least squares (green curves) modes on the values of γ . (For interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

oning, etc.), it tends to have less restrictions for the parameters of nonlocal conditions (see also [24]).

An interesting topic for the future research could be the comparison of the RBF methods and the methods based on traditional techniques (such as finite differences) applied for PDEs with nonlocal boundary conditions. Also it would be useful to investigate how successful the optimal shape parameter selection methods (e.g. [43–48]) can be in case of such nonclassical problems.

It is well-known that the spectral properties of the matrices arising after finite difference discretisation of nonlocal problems have significant influence on the applicability of the finite difference methods. For example, the spectral structure of the discretisation matrices affect the convergence of iterative methods for discrete problems solution [49]. An interesting and important question is how the spectral properties of the RBF collocation matrices depend on the parameters of nonlocal boundary conditions and how these properties influence the method.

In some sense, nonlocal multipoint boundary conditions can be interpreted as special cases of nonlocal integral conditions.

Our preliminary investigation indicates that when dealing with nonlocal conditions involving integration over an arbitrary multidimensional domain, additional problems related to the numerical integration and ill-conditioning of the method can arise. Definitely, these issues also deserve attention.

In the current paper and our previous studies [24, 26], we applied the standard Kansa’s method for the solution of nonlocal problems. However, there exists various different formulations of the Kansa method [12]. It would be interesting to investigate differences between these methods in the context of nonlocal differential problems.

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